Homotopy questions and stability of dynamical systems

Applied CATS Seminar

Wouter Jongeneel[*] (KTH (DCS, TDA) and Digital Futures)

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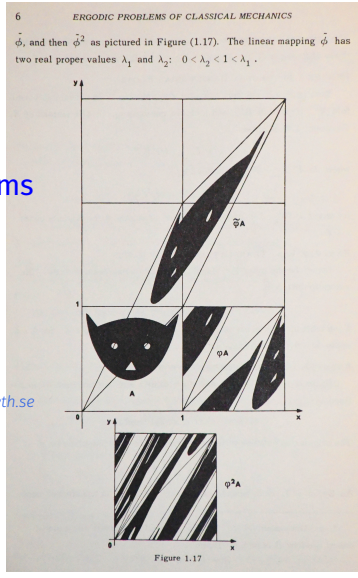
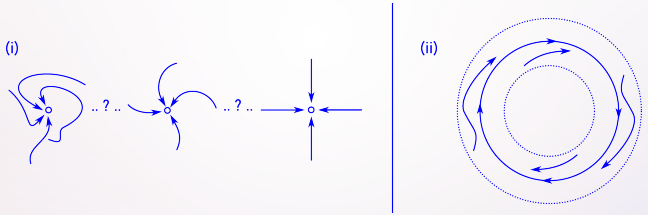
[*] Supported by Digital Futures through the project *Programmability of Cells*. Contact: *wouterjo@kth.se*



Today:

(i) Are dynamical systems that render a set *asymptotically stable*, *homotopic* through dynamical systems that preserve this notion of stability?

(ii) When is a *closed* attractor, on a metric space, homotopy equivalent to its basin of attraction?



Context?

In the beginning (e.g., see [Maw94; LP17]): Poincaré and Lyapunov.

1892 was a good year:

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First formal study into stability of motion due to Lagrange, through potential energy. (Although Lagrange stability should be called Dirichlet stability..)

Consider $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, then $(x_1, \dots, x_n, t)$ was called the **motion space**. Lyapunov speaks of **stability** when for any $\varepsilon > 0$ we can find a $\delta > 0$ such that solutions $x(t)$ starting in B_δ stay in B_ε.

Then, 0 is *Lyapunov stable* if there is a $V \in C^1(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ with $V(0) = 0$, such that $V' \leq 0$. We speak of **asymptotic stability** when $V' < 0$, i.e., $\lim_{t \to \infty} x(t) = 0$.

Further developments by Četeav, Barbashin, Krasovskiĭ, ... (inControl podcast!)

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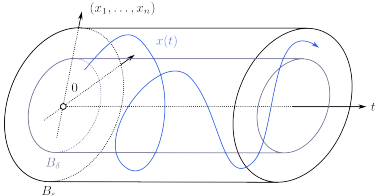

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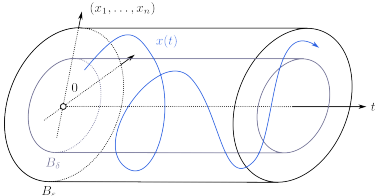

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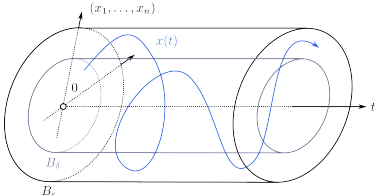

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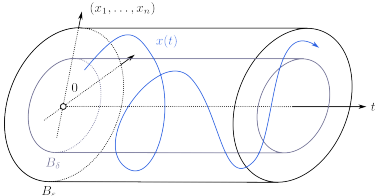

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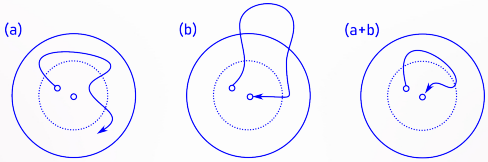

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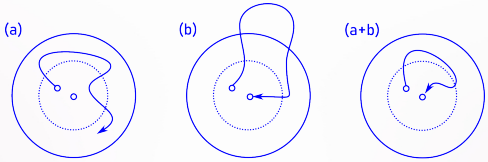

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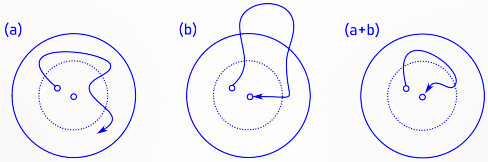

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Not only *describe* a solution to $\dot{x} = f(x)$,
but rather *prescribe* the dynamics of $\dot{x} = f(x, u)$ through a choice of input u.

A coarse view on stabilization: the Krasnosel'skiĭ-Zabreĭko (Bobylev) theorem

Theorem (Index of asymptotically stable equilibrium points [Kra68, Ch. II], [KZ84, Thm. 52.1].)

Let $p^\star \in \mathsf{M}^n$ be an isolated ***asymptotically stable*** *equilibrium point of $X \in \Gamma^r(T\mathsf{M}^n)$. Then,* $\mathrm{ind}_{p^\star}(X) = (-1)^n$.

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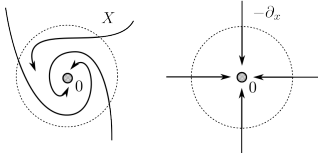
Proof sketch: In coordinates, $p^\star = 0 \in \mathbb{R}^n$, $\operatorname{ind}(\cdot)$ is a *homotopy invariant*. Details in (J. and Moulay Springer23).

$X \simeq_h$ (loc.) $-\partial_x$ ("canonical" system), without introducing new zeroes



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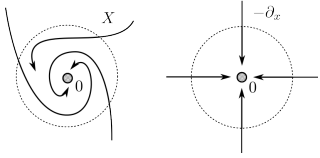

Ramification: for $\dot{x} = f(x, u)$ (C^0), a *necessary* condition for the existence of continuous, stabilizing feedback is $\exists\, \mu \in C^0 : \mathrm{ind}_0(f(\cdot, \mu(\cdot)) = (-1)^n$.

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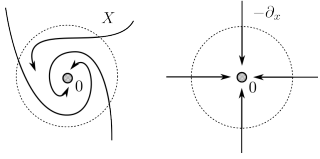

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Implies "all" well-known topological obstructions[2] (due to Zabczyk), *e.g.*, $(x, u) \mapsto f(x, u)$ must map neighbourhood $W \ni (0, 0)$ onto neighbourhood $V \ni 0$ (Brockett).

Consider the control system (topologically equivalent to a unicycle/'car'):

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2, \end{aligned} \qquad \equiv \qquad \dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{1}$$

Can we find a C^0 feedback $(x \mapsto (u_1(x), u_2(x)))$ such that 0 is stabilized?

- (1) is *controllable* as $\operatorname{span}\{g_1, g_2, [g_1, g_2]\} = \mathbb{R}^3$ [NS90]. You can park your car!
- But, Brockett's condition [Bro83] fails, no solution to $\dot{f}(x, u) = (0, 0, \varepsilon)$, $\varepsilon > 0$. So, no.

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Motivation from index theory, cont'd

"*To what extent does the homotopy index [Conley index] itself determine the equivalence class of isolated invariant sets which are related by continuation?*" [Con78, p. 83].

In general, the answer is *no* [MRS00], but we can study:

Question: "*Are dynamical systems that render a set A asymptotically stable, homotopic through dynamical systems that preserve this notion of stability?*"

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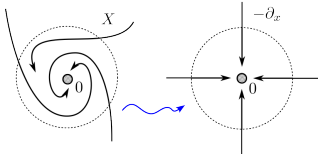

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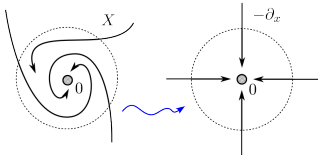

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Partial answer to (i)

Theorem (Existence of a stability preserving homotopy, WJ 26.[3])

For any two vector fields X, Y, rendering some $p^* \in M$ globally asymptotically stable (GAS), there is a homotopy between the corresponding semiflows $\varphi(\cdot; X)$ and $\varphi(\cdot; Y)$ such that p remains GAS along the homotopy.

(recently, big leap by Kvalheim [Kva25].) Crux: 'homotope' everything to $\dot{x} = -x$.

[3] WJ, "Asymptotic stability equals exponential stability—*while* you twist your eyes," 2026. Systems & Control Letters. Indeed, homage to [GSW99]: Grüne, Sontag and Wirth, "*Asymptotic stability equals exponential stability, and ISS equals finite energy gain—if you twist your eyes*," 1999. Systems & Control Letters. See also: WJ and Roland Schwan, 2024. IEEE TAC.

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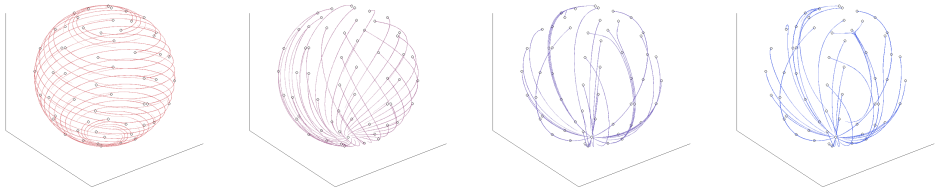

Figure 1: Such a homotopy must exist, see: https://wjongeneel.nl/images/thumbs/featured/figS.gif

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Inspired by Morse theory, the proof proceeds as:

(0) Let X be the vector field and V a C^∞ Lyapunov function, then, 'homotope' $\varphi^\tau(\cdot; X)$ to $\varphi^\tau(\cdot; -\nabla V)$, e.g., use $(1-s)X - s\nabla V$, with $s \in [0,1]$.

(1) Extend [GSW99]: show that for all $\gamma \in \mathcal{K}_\infty$,[4] there is a $T \in \mathrm{Homeo}^+(\mathbb{R}^n; \mathbb{R}^n)$ (not just $T \in \mathrm{Homeo}(\mathbb{R}^n; \mathbb{R}^n)$) such that $V(T^{-1}(x)) = \gamma(\|x\|_2)$.

(i) As $V^{-1}(c) \simeq_h \mathbb{S}^{n-1}$, there is a homeomorphism $S : V^{-1}(c) \to \mathbb{S}^{n-1}$.

(ii) $T : \mathbb{R}^n \to \mathbb{R}^n$ via $T(0) = 0$, otherwise $T(x) = \gamma^{-1}(V(x))\, S \circ \pi_c$.[5]

(iii) Now, keep S, or flip a coordinate (flip the sign of the degree).

---

[4] $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strictly monotone, C^0, $\gamma(0) = 0$ and $\lim_{s \to +\infty} \gamma(s) = +\infty$.

[5] Let ψ be the local flow corresponding to $\dot{x} = \nabla V(x) / \|\nabla V(x)\|_2^2$, fix some $c > 0$ and define $\pi_c : \mathbb{R}^n \setminus \{0\} \to V^{-1}(c)$ through $\pi_c(x) := \psi(c - V(x), x)$. See that $V(\psi(t, x)) = V(x) + t$.

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[4] $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strictly monotone, C^0, $\gamma(0) = 0$ and $\lim_{s \to +\infty} \gamma(s) = +\infty$.

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Inspired by Morse theory, the proof proceeds as:

(0) Let X be the vector field and V a C^∞ Lyapunov function, then, 'homotope' $\varphi^\tau(\cdot; X)$ to $\varphi^\tau(\cdot; -\nabla V)$, e.g., use $(1-s)X - s\nabla V$, with $s \in [0,1]$.

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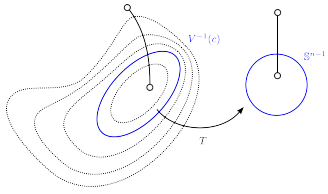

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every homotopy m-sphere is C-isomorphic to $\mathbb{S}^m$, $\mathrm{C} = \{\mathrm{Top}, \mathrm{PL}, \mathrm{Diff}\}$.

True for Top (concluded by Perelman), for Diff true for, e.g., $m = 1, 2, 3, 5, 6$, known to fail for $m = 7$ (Milnor's exotic spheres).

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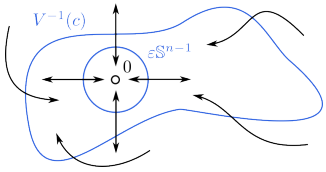

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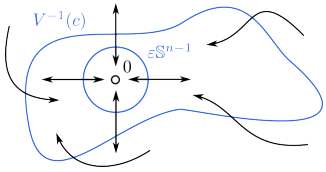

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A lot can be said about equilibria on $\mathbb{R}^n$, what about more general attractors[6]?

(ii): In particular, given an attractor A on a space X, could it be a *global* attractor?

[6] For more motivation, background and results, see: WJ and Emmanuel Moulay, "Topological Obstructions to Stability and Stabilization: History, Recent Advances and Open Problems", 2023. Springer

Lyapunov theory beyond equilibria.

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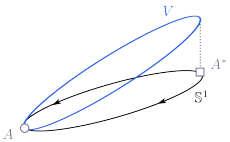

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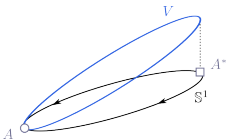

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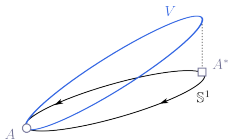

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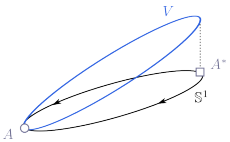

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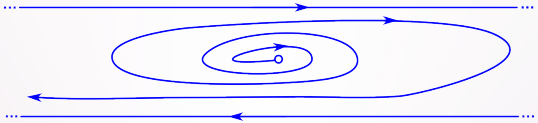
But $\omega(U) = A$? Does it always suffice?



Without compactness, what can be done?

Despite pioneering work by Bhatia[†] et al., Lyapunov theory is not 'finished' for semiflows.

Let A be an attractor* under a semiflow on X. For the following settings there is a 'complete' Lyapunov theory:

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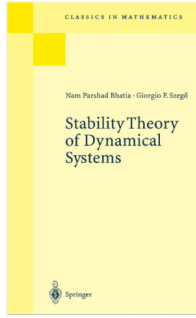

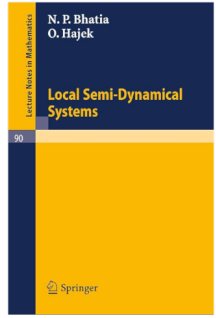

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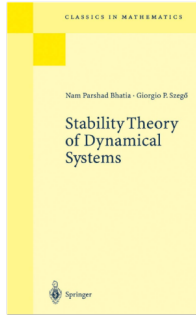

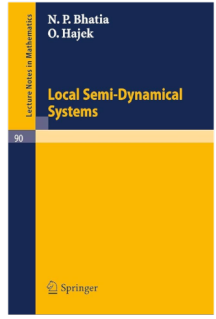

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Topological properties of *compact* attractors

Let x_* be GAS under some smooth ODE $\dot{x} = f(x)$, then, the *basin of attraction* $B(x_*)$ is a *contractible* set, e.g., there is no smooth ODE such that $q_* \in \mathbb{S}^1$ is GAS.

How to generalize beyond points, that is, to generic (asymptotically stable!) *compact* attractors?

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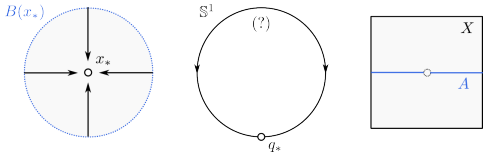

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*If a compact set A is invariant and asymptotically stable under a semiflow, on a locally compact Hausdorff space, then, A is a **weak** deformation retract of $B(A)$. (Shown using Lyapunov functions.)*

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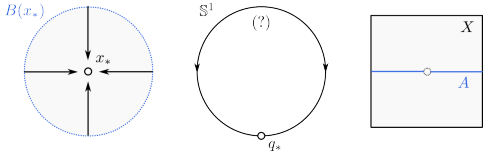

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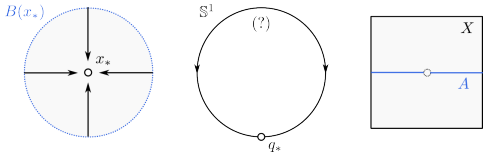

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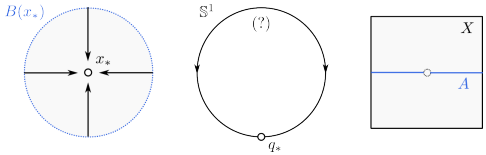

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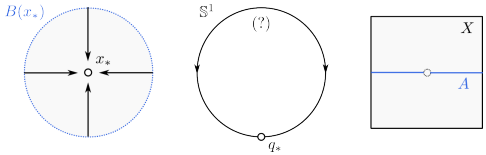

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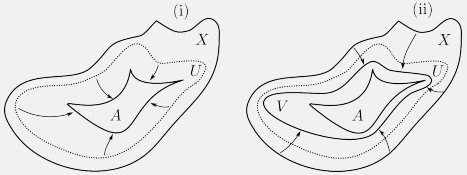
Answer (*sketch*):

When for a closed subset $A \subseteq X$ there is an open neighbourhood $U \subseteq X$ of A such that A is a "$\cdots$" retraction of U, then A is a *neighbourhood retract*, of type "$\cdots$".



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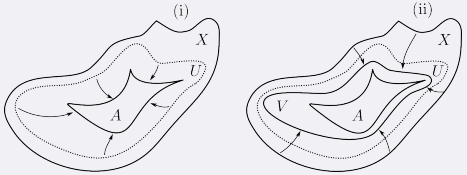

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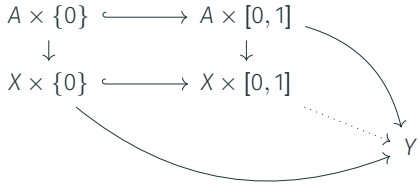
A continuous map $i : A \to X$ is said be a ***cofibration*** if the following commutative diagram can be completed for any Y:



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A pair (X, A) is a **neighbourhood deformation retract** (NDR) pair if:

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Theorem (Cofibrations and NDR pairs [May99, Ch. 6])

Let A be closed in X, then, the inclusion $\iota_A : A \hookrightarrow X$ is a cofibration $\iff$ (X, A) is an NDR pair.

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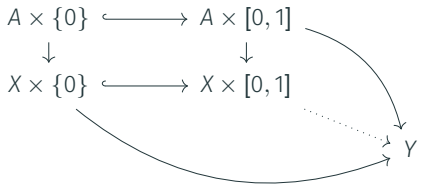

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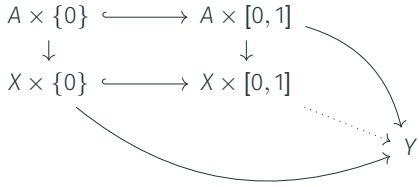

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Example (Warsaw circle, dynamics $\implies$ topology):
The Warsaw circle $\mathbb{W}^1$ can be an attractor with an annular neighbourhood $A \subset \mathbb{R}^2$ as $B(\mathbb{W}^1)$ [Has79][8]. However, $\mathbb{W}^1$ cannot be a strong deformation retract of A since $\pi_1(\mathbb{W}^1) \cong 0$ while $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Hence, $\iota_{\mathbb{W}^1} : \mathbb{W}^1 \hookrightarrow A$ *cannot* be a cofibration.

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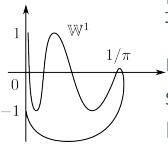

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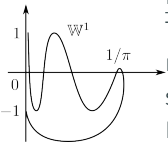

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Compact attractors are well-understood [Aki93], what about attractors that are **merely closed**?

Can we just do the same as before? No.

Lin, Yao and Cao provide an interesting closed attractor [LYC22, Ex. 22], in correcting work by Wilson. ((X, d) a metric space with $X := \mathbb{R}^2 \setminus \{(1, 0)\}$ and $A = \mathbb{S}^1 \cap X$. Could U be a metric neighbourhood?)

Here, $A \hookrightarrow B(A)$ *is* a cofibration (tubular neighbourhood), but $A \cong \{*\}$ while $B(A) \cong \mathbb{S}^1 \vee \mathbb{S}^1$.

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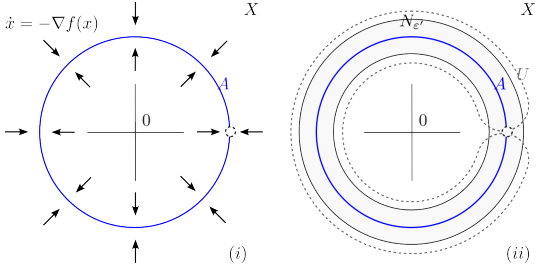

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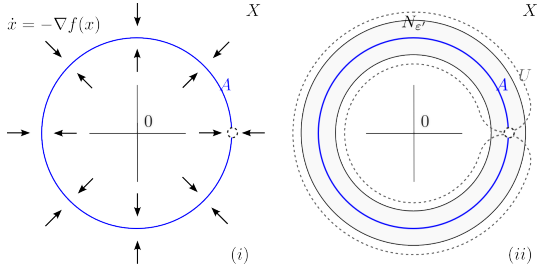

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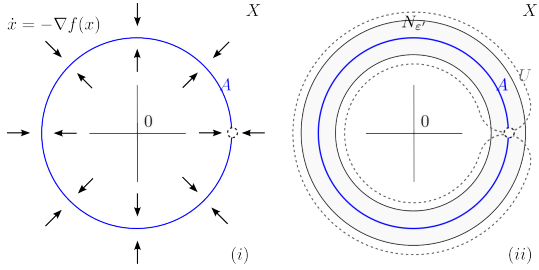

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The solution is rather technical[9]. We provide some pointers. (Heavily inspired by Auslander, Bhatia, ...)

(i) The metric and topological definitions of stability are *not* equivalent (why?).
Let $N_\varepsilon(A;d) := \{x \in X \mid d(x,A) < \varepsilon\}$.
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Combine (1) and (2): ***uniform asymptotic stability*** (see [BS70] for the Lyapunov theory).

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(i) The metric and topological definitions of stability are *not* equivalent (why?).
Let $N_\varepsilon(A;d) := \{x \in X \mid d(x,A) < \varepsilon\}$.
(1): A is ***uniformly stable*** when for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\{\varphi^t(x) \mid x \in N_{\delta(\varepsilon)}(A;d),\ t \geq 0\} \subseteq N_\varepsilon(A;d)$.
(2): A is a ***uniform attractor*** if there is some $\alpha > 0$, and for any $\varepsilon > 0$ a $T(\varepsilon) > 0$ such that $\{\varphi^t(x) \mid x \in N_\alpha(A;d),\ t \geq T(\varepsilon)\} \subseteq N_\varepsilon(A;d)$.
Combine (1) and (2): ***uniform asymptotic stability*** (see [BS70] for the Lyapunov theory).

(ii) Again, we employ NDRs, but now, the neighbourhood that deformation retracts onto A must contain some *metric* neighbourhood.

(iii) We formalize this using **filters** (inspired by Auslander† [Aus77]) and '$\mathscr{F}$-cofibrations'.

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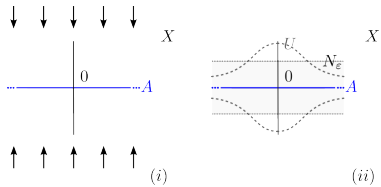

Theorem (Cofibration and closed attractors, WJ 25.[10])

Let A be a closed subset of a separable, locally compact metric space (X, d). Suppose that A is uniformly asymptotically stable, with $B(A)$ containing $N_\varepsilon(A; d)$ for some $\varepsilon > 0$. Then, A ***is a strong deformation retract of*** *$B(A)$* ***if and only if*** *$\iota_A : A \hookrightarrow B(A)$* ***is a*** *$\mathscr{F}_d$-**cofibration.***

What about a topological definition of stability? Assert stability using a *single continuous* Lyapunov function $V : X \to \mathbb{R}_{\geq 0}$? $\mathbb{R}_{\geq 0}$ is second countable. $x \mapsto d(x, A)$ allows for getting a grip on metric neighbourhoods using a *single* continuous function.

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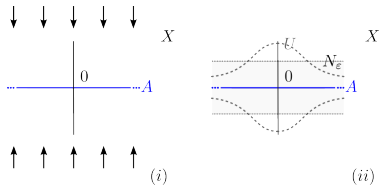

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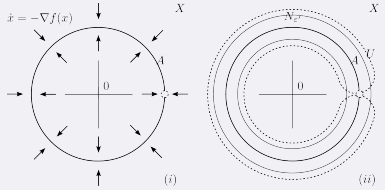
Answer to $A \not\simeq_h B_u(A)$ (circular example):



Although one can 'see $A \not\simeq_h B_u(A)$', our results tell us that for A to be a strong deformation retract of $B_u(A)$, we must have

$\iota_U \simeq_h \iota_A \circ r$,

for $r : U \to A$ a retract, U a metric neighbourhood of A and $\iota_U : U \hookrightarrow X$, $\iota_A : A \hookrightarrow X$ inclusion maps.

Then, we look at the factorization:

$$\pi_1(U) \xrightarrow{\iota_U} \pi_1(B_u(A)) = \pi_1(U) \xrightarrow{r} \pi_1(A) \xrightarrow{\iota_A} \pi_1(B_u(A))$$

which clearly fails, hence we recover the conclusion.

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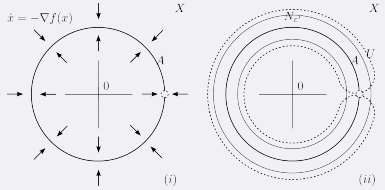

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So, even for closed attractors, we are starting to understand when $A \simeq_h B_u(A)$, can we use this?

Hybrid control and homology (as pioneered in [Bar23]).

Given these results on *strong deformation retracts*, we can *link topological properties* of A and $B(A)$.

Suppose we want to globally stabilize a set A on X, i.e., given $\dot{x} = f(x, u)$ on (X, d) find some C^0 feedback $x \mapsto \mu(x)$. 'How far is X from $B(A)$?' Can it be done, do we need *dis*continuous feedback?

Let there be a set of '*cuts*' C such that $A \hookrightarrow X \setminus C$ is a $\mathscr{F}_d$-cofibration and A is uniformly asymptotically stable on $X \setminus C$, then, $A \simeq_h X \setminus C$.

Let C be c-dimensional and regular enough, it follows from *Thom's isomorphism* that:

$$\cdots \to H^{n-c}(C; \mathbb{Z}) \to H^n(X; \mathbb{Z}) \to H^n(A; \mathbb{Z}) \to H^{n-c+1}(C; \mathbb{Z}) \to \cdots \tag{3}$$

Then, from (3), we obtain directly the following inequalities:

$$\begin{aligned} \beta^n(X) &\leq \beta^{n-c}(C) + \beta^n(A) \\ \beta^n(A) &\leq \beta^n(X) + \beta^{n-c+1}(C) \\ \beta^{n-c}(C) &\leq \beta^{n-1}(A) + \beta^n(X), \end{aligned} \tag{4}$$

where $\beta^i(\cdot) = \operatorname{rank}(H^i(\cdot; \mathbb{Z}))$, i.e., (4) understood as topological constraints on C. (☕ fun.)

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Thinking of 'canonical systems' ($\dot{x} = -x$) can help, shift the focus to transformations, not direct parametrizations (there is more Koopman-related work).

Cofibrations help in capturing homotopy equivalence of A and $B(A)$ (Did not discuss: *shape equivalence*.) Can we sharpen topological constraints on the set of cuts (include the embedding $A \hookrightarrow X$)?

Lyapunov theory has been the key tool in all results. Can we discover more transformations? More homotopy results?

Art of the state... (Sepulchre)

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Pictures

Arnold's cat map, picture from:
https://galileo-unbound.blog/2019/06/16/vladimir-arnolds-cat-map/

Nam Bhatia: *https://mathstat.umbc.edu/faculty/*

Charles Conley: *https://photos.math.wisc.edu/index.php?/category/2*

Robion Kirby: *https://celebratio.org/Kirby_RC/cover/477/*

Joeseph Auslander:
https://markauslander.com/2025/04/12/remembering-joe-auslander-1930-2025/